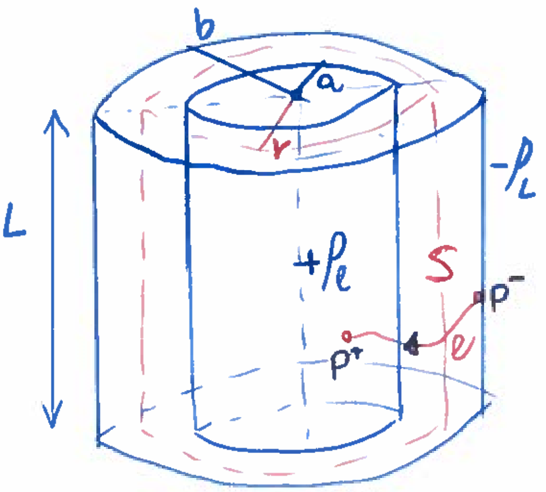
EX 1

A COAXIAL CABLE IS MADE BY TWO CYLINDRICAL CONDUCTORS OF RADIUS a (INNER) AND b (OUTER). A LINEAR CHARGE DENSITY $+\rho_\ell$ AND $-\rho_\ell$ IS DISTRIBUTED ON THE TWO CYLINDERS.
A DIELECTRIC MATERIAL (ε_r) IS PLACED BETWEEN THE CYLINDERS.

CALCULATE THE CAPACITANCE OF THE COAXIAL CABLE PER UNIT LENGTH



N.B. THE COAXIAL IS INFINITE, BUT WE CAN CONSIDER THAT HAS A LENGTH OF L.

THE CAPACITY FOR THIS SEGMENT OF COAXIAL IS $C = \dfrac{Q}{\Delta V}$

- APPLYING GAUSS LAW IN THE DIELECTRIC: $a < r < b$

$$\oint_S \vec{D} \cdot d\vec{s} = Q_{INT} \qquad \left[\begin{array}{l} \vec{D} = \varepsilon_0 \varepsilon_r \cdot \vec{E} = \varepsilon_0 \varepsilon_r E_r \vec{u} \\ d\vec{s} = ds\, \vec{u}_r \\ Q_{INT} = \rho_L \cdot L \end{array}\right.$$

$$\int_S \varepsilon_0 \varepsilon_r E_r \, ds = \varepsilon_0 \varepsilon_r E_r \cdot (2\pi r \cancel{L}) = \rho_L \cdot \cancel{L}$$

$$\boxed{E_r = \frac{\rho_\ell}{2\pi \varepsilon_0 \varepsilon_r \cdot r}}$$

- NOW WE CALCULATE THE POTENTIAL DIFFERENCE BETWEEN THE TWO CONDUCTORS ($\Delta V = V^+ - V^-$)

$$\vec{E} = E_r \, \vec{u}_r$$
$$d\vec{\ell} = (-dr)(-\vec{u}_r) = dr\, \vec{u}_r$$

$$\Delta V = V^+ - V^- = V(p^+) - V(p^-) = -\int_{p^-}^{p^+} \vec{E} \cdot d\vec{\ell} =$$

$$V(b) - V(a) = -\int_{r=b}^{a} E_r \, dr = -\int_b^a \frac{\rho_\ell}{2\pi \varepsilon_0 \varepsilon_r r} dr = \boxed{\frac{\rho_\ell}{2\pi \varepsilon_0 \varepsilon_r} \ln\left(\frac{b}{a}\right)}$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{\lambda} \cdot L}{\frac{\cancel{\lambda}}{2\varepsilon_0 \varepsilon_r \pi} \ln\left(\frac{b}{a}\right)} = \frac{2\pi \varepsilon_0 \varepsilon_r \cdot L}{\ln\left(\frac{b}{a}\right)}$$

$$\mathcal{C} = \frac{C}{L} = \boxed{\frac{2\pi \varepsilon_0 \varepsilon_r}{\ln\left(\frac{b}{a}\right)}}$$

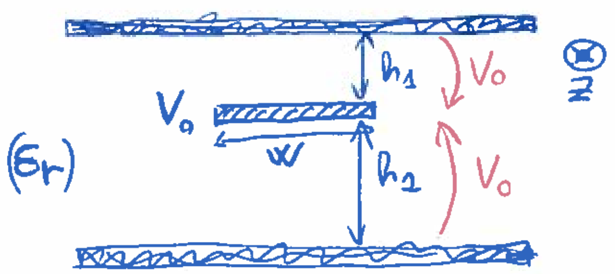
CAPACITY PER UNIT OF LENGTH

EX 2

A STRIPLINE IS MADE BY A METAL STRIP WITH A WIDTH $W = 3\,mm$ BETWEEN TWO METAL PLANES. THE METAL STRIP IS KEPT AT A POTENTIAL V_0, WHILE BOTH METAL PLANES ARE GROUNDED.

THE DISTANCES BETWEEN THE STRIP AND THE METAL PLANE ARE $h_1 = 0{,}5\,mm$ AND $h_2 = 1\,mm$.

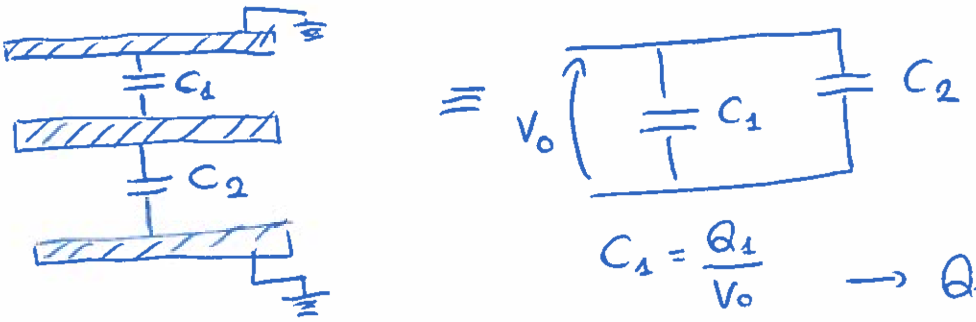
- CALCULATE THE CAPACITY PER UNIT LENGTH OF THE STRIPLINE GIVEN THAT THE SPACE BETWEEN THE PLANES IS FILLED WITH DIELECTRIC WITH $\varepsilon_r = 3$.



NOTE THAT THE STRIPLINE IS INFINITE EXTENDED ALONG z DIRECTION.

THE PLANES ARE GROUNDED: IT MEANS THAT $V = 0$ ON PLANES, SO THE POTENTIAL DIFFERENCES BETWEEN PLANES AND THE INTERNAL METAL IS V_0: WE HAVE TWO CAPACITORS WITH PARALLEL PLANES.

WE CAN DRAW THE EQUIVALENT CIRCUIT WITH THE TWO CAPACITORS:



THE TWO CAPACITORS ARE IN PARALLEL CONFIGURATION.

$$C_1 = \frac{Q_1}{V_0} \qquad C_2 = \frac{Q_2}{V_0}$$

$$\longrightarrow Q_{TOT} = Q_1 + Q_2 = C_1 \cdot V_0 + C_2 \cdot V_0 = (C_1 + C_2) \cdot V_0 = C_{eq} \cdot V_0$$

$C_{eq} = C_1 + C_2$ THE CAPACITY OF THE STRIPLINE IS DONE BY THE SUM OF THE CAPACITIES OF TWO CAPACITORS WITH PARALLEL PLANES, AND WE KNOW THAT

$$C = \frac{\varepsilon_0 \cdot \varepsilon_r \cdot A}{d}$$

$$C_1 = \frac{\varepsilon_0 \cdot \varepsilon_r \cdot (w \cdot L)}{h_1}$$

$$C_2 = \frac{\varepsilon_0 \cdot \varepsilon_r \cdot (w \cdot L)}{h_2}$$

$$\longrightarrow \quad C_{eq} = (\varepsilon_0 \varepsilon_r w L)\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$$

CAPACITY PER UNIT LENGTH

$$\mathcal{C}_{eq} = \frac{C_{eq}}{L} = \boxed{\varepsilon_0 \varepsilon_r \cdot w \left(\frac{h_1 + h_2}{h_1 \cdot h_2}\right)}$$

$\varepsilon_0 = 8,854 \cdot 10^{-12}$ F/m

$\varepsilon_r = 3$

$w = 3$ mm

$h_1 = 0,5$ mm

$h_2 = 1$ mm

WE FOUND THAT

$$C_{eq} = C_1 + C_2 = 1,59 \cdot 10^{-10} + 0,53 \cdot 10^{-10} =$$

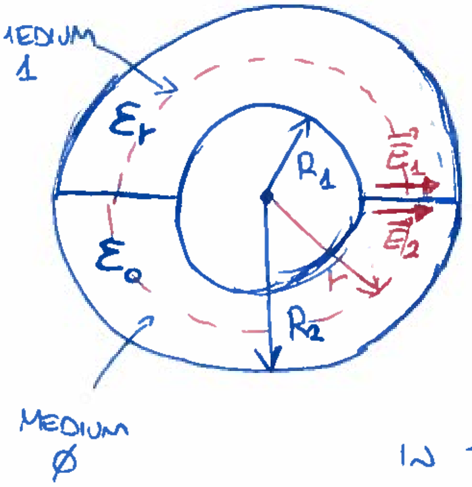
$$= \boxed{2,12 \cdot 10^{-10} \ \frac{F}{m}} = 212 \ \frac{pF}{m}$$

EX 3

A SPHERICAL CAPACITOR IS PARTIALLY FILLED WITH A DIELECTRIC WITH RELATIVE DIELECTRIC CONSTANT ε_r (SEE FIGURE)

CALCULATE THE CAPACITANCE OF THE STRUCTURE.

↳ INNER RADIUS R_1
↳ OUTER RADIUS R_2
↳ CHARGE IS Q



DUE TO THE SYMMETRY OF THE PROBLEM WE EXPECT AS USUAL THAT:

$$\vec{E} = E_r \vec{u}_r$$ (EL. FIELD DEPENDS ONLY ON THE RADIUS)

IN THE MEDIUM Ø : $D_0 = \varepsilon_0 E_0$

IN THE MEDIUM 1 : $D_1 = \varepsilon_0 \cdot \varepsilon_r \cdot E_1$

WE CAN APPLY GAUSS LAW:

$$\underset{S}{\oint} \vec{D} \cdot d\vec{S} = \underset{S}{\oint} \underbrace{\left(\vec{D}_0 + \vec{D}_1\right)}_{\text{SUPERPOSITION of EFFECTS}} \cdot d\vec{S} = \underset{S/2}{\oint} D_0\, dS + \underset{S/2}{\oint} D_1\, dS =$$

sphere with RADIUS $R_1 < r < R_2$

$$= \boxed{D_0 \frac{4\pi r^2}{2} + D_1 \cdot \frac{4\pi r^2}{2} = Q}$$

AT THE INTERFACE BETWEEN THE TWO MEDIA (AIR ⓪ AND DIELECTRIC ①) WE CAN APPLY THE CONTINUITY CONDITIONS.

WE KNOW THAT THE TANGENT COMPONENTS OF THE ELECTRIC FIELD IS CONSERVATIVE

$$\boxed{E_{0r} = E_{1r}} \longrightarrow E_0 = E_1$$

$$\downarrow$$

$$\frac{D_0}{\cancel{\varepsilon_0}} = \frac{D_1}{\cancel{\varepsilon_0}\varepsilon_r}$$

$$\boxed{D_0 = \frac{D_1}{\varepsilon_r}}$$

FROM BOTH CONDITIONS:

$$\begin{cases} Q = D_0 2\pi r^2 + D_1 2\pi r^2 \\ D_1 = D_0 \cdot \varepsilon_r \end{cases} \longrightarrow Q = D_0 2\pi r^2 + \varepsilon_r D_0 2\pi r^2$$

WE FIND:

$$\begin{cases} D_0 = \dfrac{Q}{(1+\varepsilon_r)\cdot 2\pi r^2} \\ D_1 = \dfrac{Q\varepsilon_r}{(1+\varepsilon_r) 2\pi r^2} \end{cases} \longrightarrow \begin{cases} E_0 = \varepsilon_0 D_0 = \dfrac{Q}{2\pi r^2} \cdot \dfrac{1}{\varepsilon_0(1+\varepsilon_r)} \\ E_1 = \varepsilon_0 \varepsilon_r \cdot D_1 = \dfrac{Q}{2\pi r^2} \cdot \dfrac{1}{\varepsilon_0(1+\varepsilon_r)} \end{cases}$$

OBVIOUSLY WE FIND $E_0 = E_1$ (WE IMPOSED IT VIA BOUNDARY CONDITIONS)

WE CAN CALCULATE POTENTIAL DIFFERENCE:

$$\Delta V = V(R_1) - V(R_2) = -\int_{R_2}^{R_1} \vec{E} \cdot d\vec{r} = -\int_{R_2}^{R_1} \frac{Q}{2\pi\varepsilon_0(1+\varepsilon_r)} r^{-2} dr =$$

$$= -\frac{Q}{2\pi\varepsilon_0(1+\varepsilon_r)} \int_{R_1}^{R_2} r^{-2} = -\frac{Q}{2\pi\varepsilon_0(1+\varepsilon_r)} \left[-\frac{1}{r}\right]_{R_1}^{R_2} =$$

$$= \frac{Q}{2\pi\varepsilon_0(1+\varepsilon_r)} \left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

THE CAPACITANCE IS:

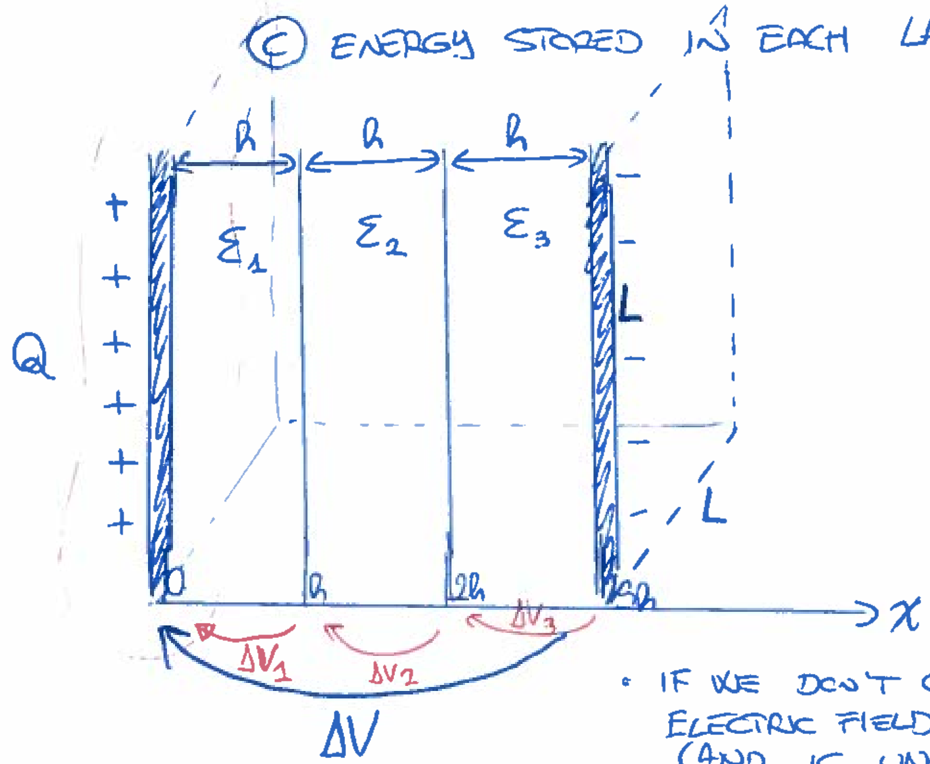
$$C = \frac{Q}{\Delta V} = \boxed{2\pi\varepsilon_0(1+\varepsilon_r)\left(\frac{R_1 \cdot R_2}{R_2 - R_1}\right)}$$

EX 4

A PARALLEL PLATE CAPACITOR ($L \times L$ PLATES) IS FILLED WITH 3 DIELECTRICS (ε_{r_1}, ε_{r_2} AND ε_{r_3}) WITH THICKNESS h (EACH).

A POTENTIAL DIFFERENCE ΔV IS APPLIED TO THE PLATES OF THE CAPACITOR.

- CALCULATE
  - (a) ELECTRIC FIELD
  - (b) CAPACITANCE
  - (c) ENERGY STORED IN EACH LAYER

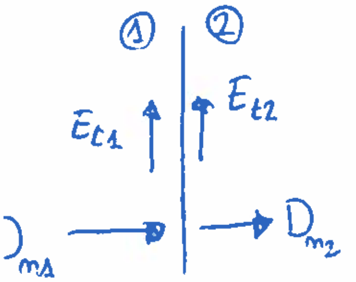


GIVEN THAT THERE IS A POTENTIAL DIFFERENCE BTW PLATES, IT MEANS THAT CHARGES ARE ACCUMULATED ON THEM.

TOTAL CHARGE ON POSITIVE PLATE IS Q.

- IF WE DON'T CONSIDER THE SIDE EFFECTS, THE ELECTRIC FIELD IS GOING FROM ⊕ TO ⊖ (AND IS UNIFORM).

WE CAN APPLY AT THE INTERFACES OF DIELECTRICS THE CONTINUITY CONDITIONS:



$\cancel{E_{t1} = E_{t2}}$

$D_{m2} - D_{m1} = \cancel{\sigma}$

DENSITY CHARGE AT THE INTERFACES. IN THIS CASE ARE NOT CHARGED

IN THIS CASE THE ELECTRIC FIELD IS NORMAL TO THE INTERFACES, SO

$$D_{m1} = D_{m2} = D_{m3}$$

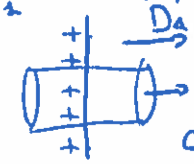
$$\boxed{D_1 = D_2 = D_3}$$

WE HAVE 3 REGIONS I, II AND III

APPLYING GAUSS LAW IN THE 1^{ST} MEDIUM:

$$\oint_S \vec{D}_1 \cdot d\vec{s} = Q$$

$\vec{D}_1 = D_1 \vec{u}_x$



$d\vec{s} = ds\,\vec{u}_x$

$$\oint_S D_1\, ds = Q$$

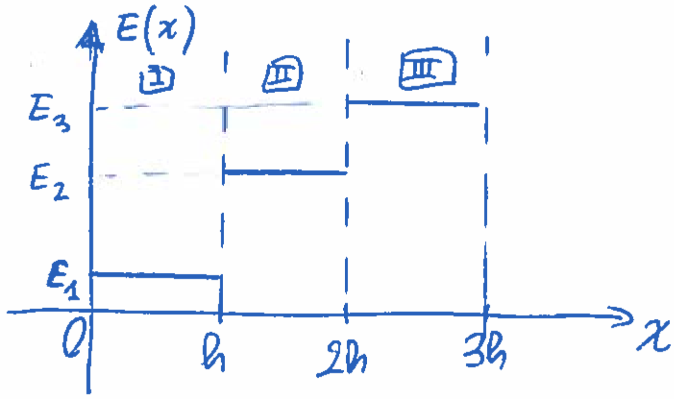
$$D_1 \cdot L^2 = Q$$

$$\boxed{D_1 = \frac{Q}{L^2}} = D_2 = D_3$$

$$\begin{cases} E_1 = \dfrac{D_1}{\varepsilon_0 \varepsilon_{r_1}} = \dfrac{Q}{\varepsilon_0 \cdot \varepsilon_{r_1} \cdot L^2} \\[2ex] E_2 = \dfrac{D_2}{\varepsilon_0 \cdot \varepsilon_{r_2}} = \dfrac{Q}{\varepsilon_0 \cdot \varepsilon_{r_2} \cdot L^2} \\[2ex] E_3 = \dfrac{D_3}{\varepsilon_0 \cdot \varepsilon_{r_3}} = \dfrac{Q}{\varepsilon_0 \cdot \varepsilon_{r_3} \cdot L^2} \end{cases}$$

IF WE ASSUME THAT

$$\varepsilon_{r_1} > \varepsilon_{r_2} > \varepsilon_{r_3}$$



THEN WE CAN CALCULATE THE POTENTIAL BETWEEN THE INTERFACES...

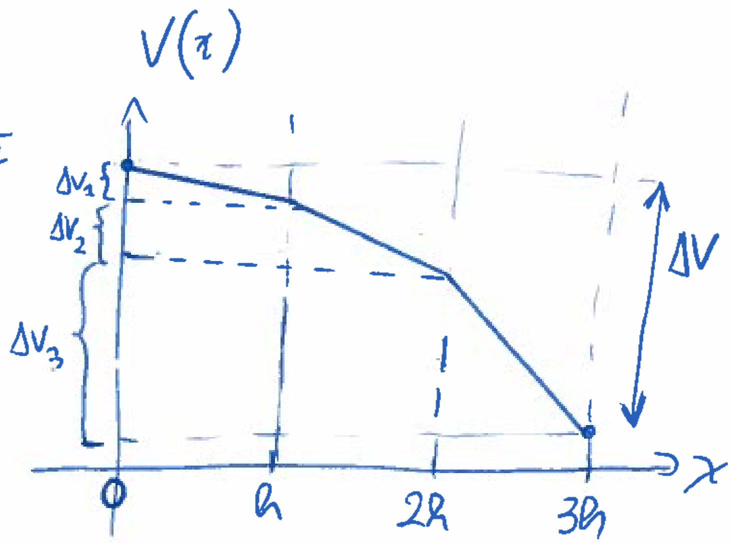
$$\Delta V_1 = V(x=0) - V(x=h) = -\int_h^0 E_1\, dx = -\int_h^0 \frac{Q}{\varepsilon_0 \varepsilon_{r_1} L^2}\, dx =$$

$$= \frac{Q \cdot h}{\varepsilon_0 \varepsilon_{r_1} \cdot L^2}$$

IN THE SAME WAY WE GET:

$$\Delta V_2 = -\int_{2h}^{h} E_2\, dx = \frac{Q \cdot h}{\varepsilon_0 \varepsilon_{r_2} \cdot L^2}$$

$$\Delta V_3 = \frac{Q \cdot h}{\varepsilon_0 \cdot \varepsilon_{r_3} \cdot L^2}$$



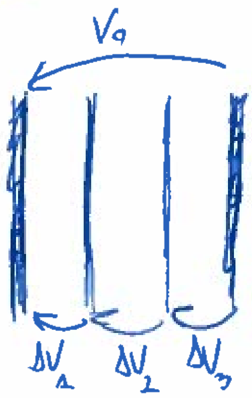
WE SEE THAT:

$$\Delta V_1 < \Delta V_2 < \Delta V_3$$

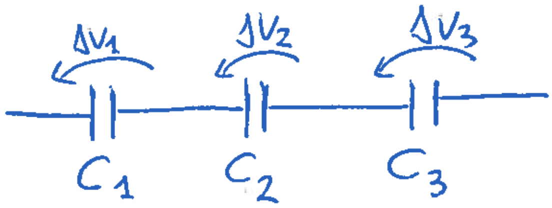
AND REMEMBER THAT POTENTIAL IS ALWAYS CONTINUOUS...

2) CAPACITANCE

We see that $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$



$\equiv$ EQUIV. CIRCUIT



SERIES CONFIGURATION
$Q_1 = Q_2 = Q_3 = Q$

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 =$$

$$= \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) =$$

$$= \frac{Q}{C_{eq}}$$

$$\boxed{C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}}$$

$$C_i = \varepsilon_0 \cdot \varepsilon_{r_i} \cdot \frac{L^2}{h}$$

CAPACITANCE OF EQUIVALENT CAPACITOR WITH // PLANES WITH DIFF. POTENTIAL ΔV_i

(IT COMES FROM $C = \varepsilon_0 \varepsilon_r \frac{A}{d}$)

3) ELECTROSTATIC ENERGY STORED IN EACH LAYER IS:

$$W_i = \frac{1}{2} Q_i \Delta V_i$$

$$W_1 = \frac{1}{2} Q_1 \cdot \Delta V_1 = \frac{1}{2} \cdot Q \cdot \Delta V_1 = \frac{1}{2} Q \left[\frac{Q}{L^2} \frac{h}{\varepsilon_0 \varepsilon_r}\right] = \frac{1}{2} \frac{Q^2}{L^2} \cdot \frac{h}{\varepsilon_0 \varepsilon_{r_1}}$$

(WE JUST GET THAT: $Q = \Delta V \cdot C_{eq}$)

$$= \frac{1}{2} \cdot \frac{\Delta V^2 \cdot C_{eq}^2}{L^2} \cdot \frac{h}{\varepsilon_0 \cdot \varepsilon_{r_1}} = \boxed{\frac{1}{2} \frac{\Delta V^2 \cdot L^2 \cdot \varepsilon_0}{h \cdot \varepsilon_{r_1} \left(\frac{1}{\varepsilon_{r_1}} + \frac{1}{\varepsilon_{r_2}} + \frac{1}{\varepsilon_{r_3}}\right)^2}}$$

WE EXPRESS Q AS FUNCTION OF ΔV because Q ...

ALTERNATIVELY WE CAN CALCULATE THE ENERGY USING ITS DEFINITION:

$$W_i = \frac{1}{2} \int_{Volume} (\vec{E}_i \cdot \vec{D}_i)\, dv$$

$$W_1 = \frac{1}{2} \int_V E_1 \cdot D_1 \, dv = \frac{1}{2} \int \underbrace{\left(\frac{Q}{L^2} \cdot \frac{1}{\varepsilon_0 \cdot \varepsilon_{r1}}\right)}_{E_1} \cdot \underbrace{\left(\frac{Q}{L^2}\right)}_{D_1} dV =$$

$$= \frac{1}{2} \cdot \frac{Q^2}{L^4} \cdot \frac{1}{\varepsilon_0 \, \varepsilon_{r1}} \cdot L^2 \cdot h =$$

$$= \frac{1}{2} \cdot \frac{Q^2}{L^2} \cdot \frac{h}{\varepsilon_0 \, \varepsilon_{r1}}$$

(AS FOUND before...)