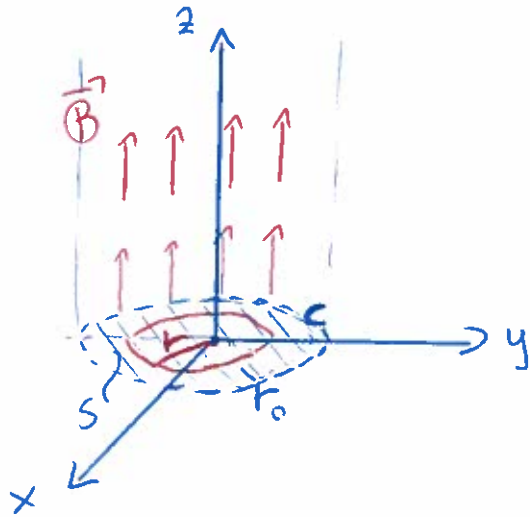


## EX 3 (FARADAY LAW)

CALCULATE THE ELECTRIC FIELD  $\vec{E}$  INDUCED BY A TIME-VARYING MAGNETIC FIELD  $\vec{B}$  GIVEN BY:

$$\vec{B} = \begin{cases} r \cdot \sin(\omega t) \vec{u}_z & r \leq r_0 \\ \emptyset & r > r_0 \end{cases}$$



WE KNOW THAT A TIME-VARYING MAGNETIC FIELD IS INDUCING AN ELECTROMOTIVE FORCE e.m.f THAT IS EQUAL TO THE CIRCULATION OF  $\vec{E}$  OVER A CLOSED LINE.



FARADAY LAW:

$$\text{e.m.f} = \oint_C \vec{E} \cdot d\vec{e} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

C: CLOSED LOOP

S: SURFACE ENCLOSED INTO THE LOOP C

THIS "FORCE" COUNTERBALANCE THE VARIATION OF THE FLUX  $\vec{B}$ , GENERATING ANOTHER MAGNETIC FIELD THAT VARIES OVER THE TIME IN THE OPPOSITE SENSE.

$$\underbrace{\oint_C \vec{E} \cdot d\vec{e}}_{\text{I}} = - \underbrace{\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}_{\text{II}}$$

WE START FROM II MEMBER OF FARADAY EQUATION; DISTINGUISHING THE CASE IN WHICH  $r \leq r_0$  AND  $r > r_0$ .

SURFACE ELEMENT  $d\vec{S} = ds \vec{u}_z = (2\pi r dr) \vec{u}_z$

→  $r < r_0$   $-\int_{S_r} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = - \int_{S_r} \frac{d}{dt} [r \sin(\omega t)] \vec{u}_z \cdot (2\pi r dr) \vec{u}_z =$

$$= - \int_{S_r} r \omega \cos(\omega t) \cdot 2\pi r dr = - \int_0^r 2\pi \omega \cos(\omega t) \cdot r^2 dr =$$

$$= - 2\pi \omega \cos(\omega t) \int_0^r r^2 dr = \boxed{- 2\pi \omega \cdot \frac{r^3}{3} \cos(\omega t)}$$

→  $r > r_0$  SINCE THE FIELD  $\vec{B}$  VARYS WITH TIME ONLY FOR  $r \leq r_0$ , THE CONTRIBUTION TO THE FLUX OF  $\frac{\partial \vec{B}}{\partial t}$  IS GIVEN ONLY BY THE FLUX CALCULATED FOR  $r \leq r_0$ :

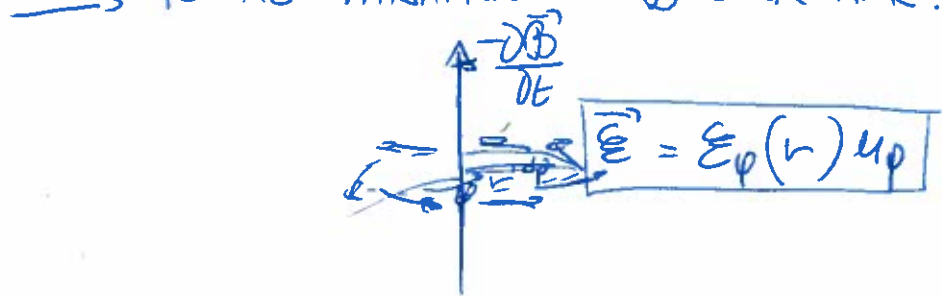
$$- \int_{S_{r_0}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = - \int_0^{r_0} 2\pi \omega \cos(\omega t) r^2 dr = \boxed{- 2\pi \omega \cdot \frac{r_0^3}{3} \cos(\omega t)}$$

I] NOW FOR THE FIRST MEMBER OF THE FARADAY LAW WE NEED TO CALCULATE THE CIRCULATION OF  $\vec{E}$  OVER THE LOOP WITH RADIUS  $r$ .

REMEMBER THAT FARADAY LAW CAN BE WRITTEN AS FOLLOW:

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

IT MEANS THAT THE INDUCED FIELD  $\vec{E}$  HAS FIELD LINES THAT ARE CLOSED AND PERPENDICULAR TO THE VARIATION OF  $\vec{B}$  OVER TIME:



SO:

$$\oint_C \vec{E} \cdot d\vec{e} = \oint_C (E_\phi \vec{u}_\phi) \cdot (r d\phi \vec{u}_\phi) = \int_0^{2\pi} E_\phi r d\phi = E_\phi r \cdot 2\pi$$

EQUATING THE TWO MEMBERS OF FARADAY LAW WE GET:

$$\boxed{r < r_0}$$

$$\mathcal{E}_\varphi \cdot r \cdot 2\pi = -2\pi \omega \frac{r^{\frac{3}{2}}}{3} \cos(\omega t)$$

$$\mathcal{E}_\varphi = -\frac{\omega r^2}{3} \cos(\omega t)$$

$$\boxed{r > r_0}$$

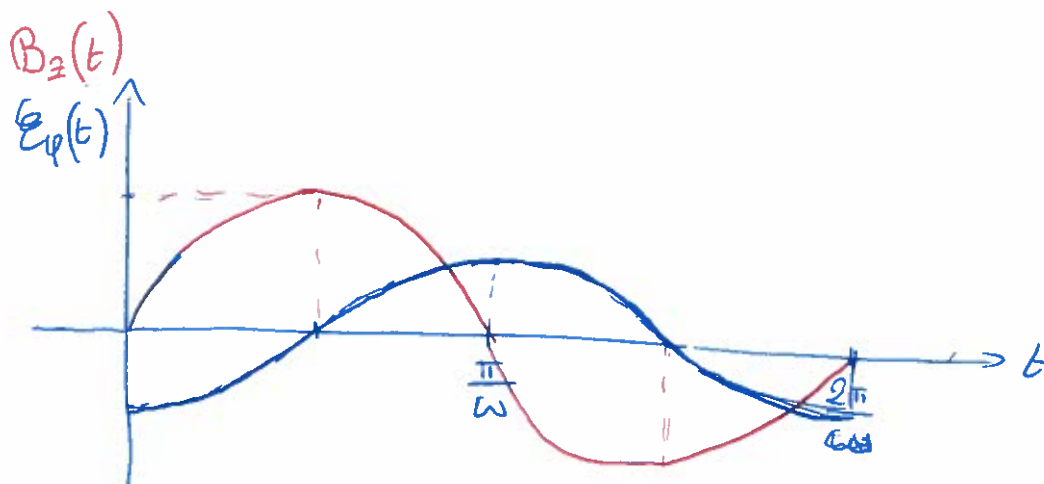
$$\mathcal{E}_\varphi \cdot r \cdot 2\pi = -2\pi \omega \frac{r_0^3}{3} \cos(\omega t)$$

$$\mathcal{E}_\varphi = -\frac{\omega r_0^3}{3r} \cos(\omega t)$$

$$\vec{\mathcal{E}} = \begin{cases} -\frac{\omega r^2}{3} \cos(\omega t) \vec{u}_\varphi \\ -\frac{\omega r_0^3}{3r} \cos(\omega t) \vec{u}_\varphi \end{cases}$$

$$\omega = 2\pi f \quad f = \frac{\omega}{2\pi}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$



WE FOUND THAT WHEN

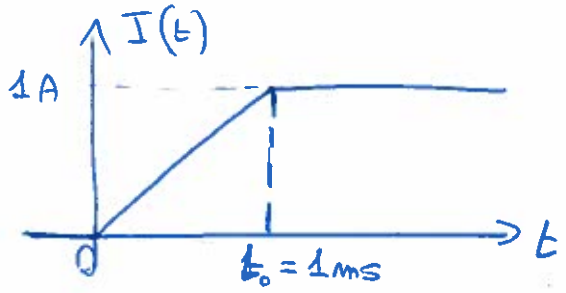
$$\left. \begin{aligned} \frac{\partial B}{\partial t} > 0 &\implies \vec{\mathcal{E}} \text{ IS NEGATIVE} \\ \frac{\partial B}{\partial t} < 0 &\implies \vec{\mathcal{E}} \text{ IS POSITIVE} \end{aligned} \right\}$$

THIS BECAUSE C.M.F  
GENERATES A MAGNETIC FLUX  
THAT COUNTERACTS THE  
VARIATION OF ORIGINAL  
MAGNETIC FLUX

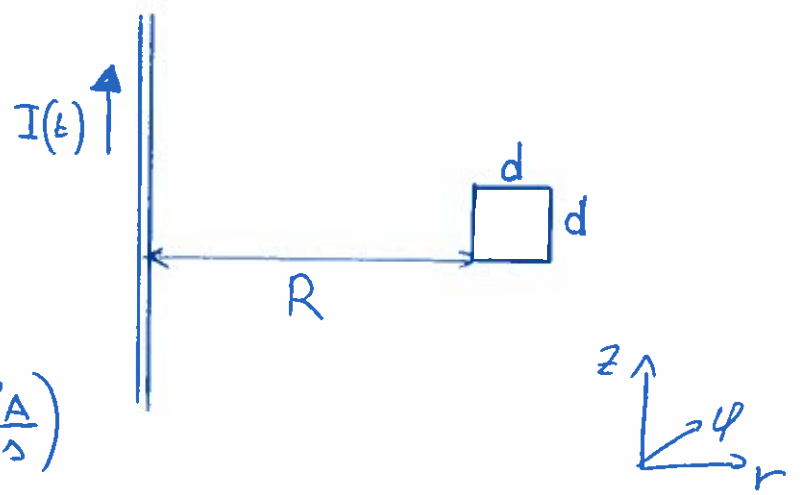
EX 4

A STRAIGHT ELECTRIC WIRE CARRYING A CURRENT  $I(t)$  IS PLACED AT DISTANCE  $R = 1 \text{ cm}$  FROM A SMALL SQUARE COIL WITH SIZE  $d = 1 \text{ mm}$ .

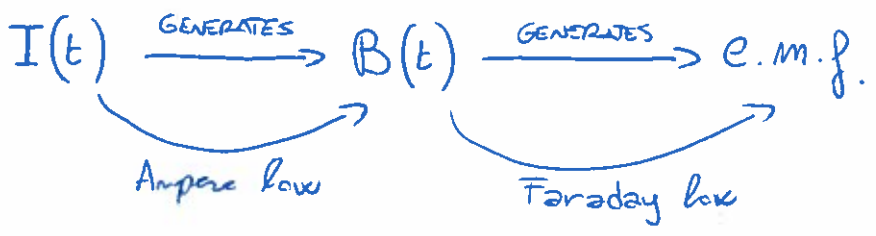
CALCULATE THE emf INDUCED IN THE COIL IF  $I(t)$  IS GIVEN BY:



$$I(t) = \begin{cases} k_0 \cdot t & t \leq t_0 \\ 1 \text{ A} & t > t_0 \end{cases} \quad \left( k_0 = 10^3 \frac{\text{A}}{\text{s}} \right)$$



WE CAN "DECOMPOSE" THE PROBLEM INTO TWO STEPS:



THE TIME-VARYING CURRENT, FLOWING THROUGH THE WIRE, IS GENERATING A TIME-VARYING MAGNETIC FIELD  $\vec{H}(t)$ , THAT FOR SYMMETRICAL REASONS IS:  $\vec{H}(t) = H_\phi(r) \vec{u}_\phi$

Ampere law  $\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S}$

$$H_\phi \cdot 2\pi \cdot R = I(t)$$

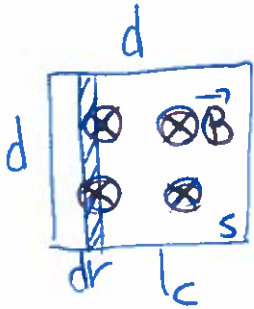
$$\vec{H}(t, R) = \frac{I(t)}{2\pi R} \vec{u}_\phi$$

SINCE THE COIL IS MUCH SMALLER THAN  $R$  ( $d \ll R$ ) WE CAN APPROXIMATE THAT THE MAGNETIC FIELD  $\vec{H}$  THAT FILLS THE COIL AREA IS CONSTANT  $\forall t$ .

$$\vec{B} = \mu \vec{H} = \frac{\mu I(t)}{2\pi R} \vec{u}_\varphi$$

SO WE FOUND THAT THE MAGNETIC FLUX DENSITY  $\vec{B}$  IS ENTERING INTO THE COIL PLANE.

FROM "Faraday law"



$$e.m.f. = \oint_C \vec{E} \cdot d\vec{e} = - \int_S \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{S}$$

$$d\vec{S} = ds(+\vec{u}_\varphi) = d \cdot dr \vec{u}_\varphi$$

$$e.m.f. = - \int_S \left( \frac{\partial B_\varphi}{\partial t} \vec{u}_\varphi \right) \cdot (d^2 \vec{u}_\varphi) = - \int_0^d \frac{\partial}{\partial t} \left[ \frac{\mu I(t)}{2\pi R} \right] \cdot d \cdot dr =$$

$$= - \left[ \frac{\partial I(t)}{\partial t} \right] \cdot \frac{\mu \cdot d^2}{2\pi R} =$$

$$= \begin{cases} - \frac{\partial}{\partial t} [k_0 t] \cdot \frac{\mu d^2}{2\pi R} = - \frac{k_0 \mu \cdot d^2}{2\pi R} & (t < t_0) \\ \emptyset & (t > t_0) \end{cases}$$

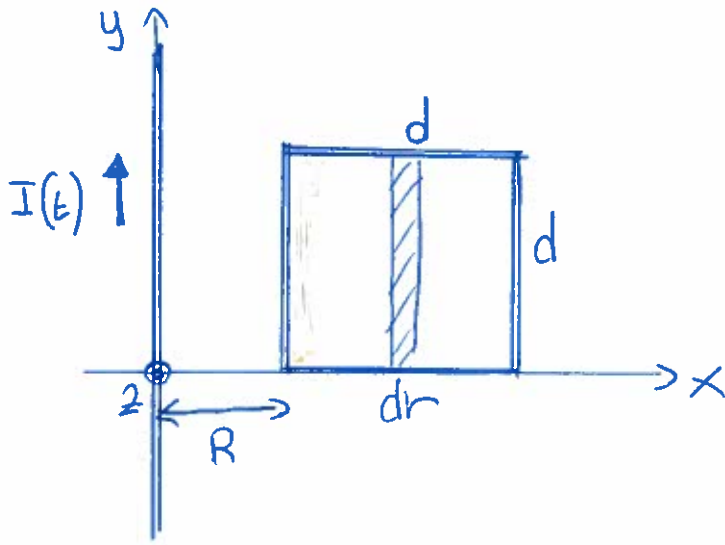
If  $t < t_0$  we get:

$$e.m.f. = 10^3 \frac{A}{s} \cdot \frac{2}{4\pi} \cdot 10^{-7} \frac{H}{m} \cdot \frac{(10^{-3} m)^2}{2\pi \cdot 10^{-2} m} =$$

$$= \frac{10^3 \cdot 2 \cdot 10^{-7} \cdot 10^{-6}}{10^{-2}} \cdot \frac{A}{s} \cdot \frac{H}{m} \cdot \frac{m^2}{m} = 2 \cdot 10^{-8} \frac{A}{s} \cdot \frac{V \cdot s}{A} = \boxed{2 \cdot 10^{-8} V}$$

# EX 5

CALCULATE THE EMF INDUCED BY A METALLIC WIRE CARRYING A CURRENT  $I(t) = 0,5 \cos(\omega t)$  [A] ON A SQUARE COIL ( $d = 20 \text{ cm}$ ) PLACED AT DISTANCE  $R = 5 \text{ cm}$  FROM THE WIRE.

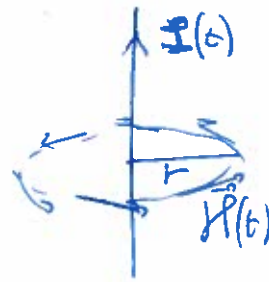


$$\begin{pmatrix} \omega = 2\pi f \\ f = 5 \text{ kHz} \end{pmatrix}$$

THE PROCEDURE FOLLOWED IS IDENTICAL TO THE ONE OF PREVIOUS EXERCISE, BUT NOW THE FIELD  $\vec{B}$  GENERATED BY  $I(t)$  IS DIFFERENT ON THE VARIOUS POINTS OF THE COIL (BECAUSE  $d > R$ ).

APPLYING AMPERE LAW:

$$\oint_C \vec{H} \cdot d\vec{e} = \int_S \vec{J} \cdot d\vec{s}$$



$$\vec{H} = H_\varphi \vec{u}_\varphi$$

$$d\vec{e} = r \cdot d\varphi \vec{u}_\varphi$$

$$H_\varphi \cdot r \cdot 2\pi = I(t)$$

$$\vec{H}(t) = \frac{I(t)}{2\pi r} \vec{u}_\varphi$$

$$\vec{B}(t) = \frac{\mu I(t)}{2\pi r} \vec{u}_\varphi$$

$$= -\frac{\mu I(t)}{2\pi r} \vec{u}_z$$

WE APPLY THE FARADAY LAW:

$$e_{\text{mf}} = \oint_C \vec{E} \cdot d\vec{e} = - \oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad d\vec{S} = ds \vec{u}_z = dr \cdot d\vec{u}_z$$

$$e_{\text{mf}} = - \oint_S \frac{\partial}{\partial t} \left( \frac{\mu I(t)}{2\pi r} \vec{u}_z \right) \cdot (d \cdot dr \vec{u}_z) =$$

$$= \int_{r=R}^{R+d} \frac{\mu \cdot d}{2\pi} \frac{\partial I(t)}{\partial t} \cdot \frac{dr}{r} =$$

$$= \frac{\mu \cdot d}{2\pi} \frac{\partial I(t)}{\partial t} \int_R^{R+d} \frac{dr}{r} =$$

$$= \frac{\mu \cdot d}{2\pi} \frac{\partial I(t)}{\partial t} \left[ \ln(r) \right]_R^{R+d} =$$

$$= \frac{\mu d}{2\pi} \ln\left(\frac{R+d}{R}\right) \frac{\partial}{\partial t} \left[ 0,5 \cdot \cos(\omega t) \right] =$$

$$= \frac{\mu d}{2\pi} \ln\left(\frac{R+d}{R}\right) \cdot \left[ 0,5 \cdot \omega \cdot -\sin(\omega t) \right]$$

$$|e_{\text{mf}}| = \frac{4\pi \cdot 10^{-7}}{2\pi} \left[ \frac{\text{H}}{\text{m}} \right] \cdot 0,2 \text{ m} \cdot \ln\left(\frac{25}{5}\right) \cdot 0,5 \cdot 2\pi \cdot 5 \cdot 10^3 [\text{Hz}] \cdot \sin(\omega t)$$

$$\mu = 4\pi \cdot 10^{-7} \frac{\text{H}}{\text{m}}$$

$$R = 5 \text{ cm}$$

$$d = 20 \text{ cm}$$

$$\omega = 2\pi \cdot 5 \text{ kHz}$$

$$= 2 \cdot 10^{-7} \cdot 0,2 \cdot \ln(5) \cdot 0,5 \cdot \overbrace{2\pi \cdot 5 \cdot 10^3}^{\omega} \sin(\omega t) \frac{\text{H} \cdot \text{m} \cdot \text{A}}{\text{m} \cdot \text{s}}$$

$$= 1,01 \cdot 10^{-3} \sin(\omega t) [\text{V}] =$$

$$\frac{\text{V} \cdot \text{s} \cdot \text{A}}{\text{A} \cdot \text{s}}$$

$$= \boxed{1,01 \cdot \sin(\omega t)} [\text{mV}]$$