EX 1

GIVEN THE THREE VECTORS:

$$\vec{A} = \vec{u}_x + 2\vec{u}_y - \vec{u}_z$$

$$\vec{B} = -2\vec{u}_x + 3\vec{u}_y + 2\vec{u}_z$$

$$\vec{C} = 3\vec{u}_x - \vec{u}_y - \vec{u}_z$$

DEMONSTRATE THAT $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$, SO THAT THE ASSOCIATIVE PROPERTY IS <u>NOT</u> VALID FOR CROSS PRODUCT.

We know that cross product doesn't have the commutative property, so we need to follow the bracket order:

- $$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ -2 & 3 & 2 \\ 3 & -1 & -1 \end{vmatrix} = \vec{u}_x(-3+2) - \vec{u}_y(2-6) + \vec{u}_z(2-9) = -\vec{u}_x + 4\vec{u}_y - 7\vec{u}_z$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ 1 & 2 & -1 \\ -1 & 4 & -7 \end{vmatrix} = \vec{u}_x(-14+4) - \vec{u}_y(-7-1) + \vec{u}_z(4+2) = \boxed{-10\vec{u}_x + 8\vec{u}_y + 6\vec{u}_z}$$

- $$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ 1 & 2 & -1 \\ -2 & 3 & 2 \end{vmatrix} = \vec{u}_x(4+3) - \vec{u}_y(2-2) + \vec{u}_z(3+4) = 7\vec{u}_x + 7\vec{u}_z$$

- $$(\vec{A} \times \vec{B}) \times \vec{C} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ 7 & 0 & 7 \\ 3 & -1 & -1 \end{vmatrix} = \vec{u}_x \cdot 7 - \vec{u}_y(-7-21) + \vec{u}_z(-7) = \boxed{7\vec{u}_x + 28\vec{u}_y - 7\vec{u}_z}$$

So we see that $-10\vec{u}_x + 8\vec{u}_y + 6\vec{u}_z \neq 7\vec{u}_x + 28\vec{u}_y - 7\vec{u}_z$

EX 2

GIVEN A SCALAR FIELD $V = x^2y + xy^2 + xz^2$, FIND THE GRADIENT OF V AND CALCULATE ITS VALUE IN P (1,-1,2)

BY DEFINITION WE KNOW THAT:

$$\nabla V = grad(V) = \frac{\partial V}{\partial x}\vec{u}_x + \frac{\partial V}{\partial y}\vec{u}_y + \frac{\partial V}{\partial z}\vec{u}_z$$

Please remind that the gradient of a scalar is a VECTOR!
The gradient of a field is equal to the sum of the derivatives of the field with respect to the three axes x, y and z.

$$\nabla V = \frac{\partial(x^2y + xy^2 + xz^2)}{\partial x}\vec{u}_x + \frac{\partial V}{\partial y}\vec{u}_y + \frac{\partial V}{\partial z}\vec{u}_z =$$

$$= (2xy + y^2 + z^2)\vec{u}_x + (x^2 + 2xy)\vec{u}_y + (2xz)\vec{u}_z$$

$$\nabla V|_P = (2\cdot 1\cdot -1 + 1 + 4)\vec{u}_x + (1 - 2)\vec{u}_y + (4)\vec{u}_z =$$

$$= 3\vec{u}_x - \vec{u}_y + 4\vec{u}_z$$

EX 3

GIVEN A VECTOR $\vec{E} = (3x^2)\vec{u}_x + (2z)\vec{u}_y + (x^2z)\vec{u}_z$, FIND THE DIVERGENCE AND CALCULATE ITS VALUE IN P (2,-2,0)

BY DEFINITION WE KNOW THAT the DIVERGENCE OF A VECTOR IS A SCALAR FIELD THAT IS CALCULATED:

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

as the sum of the partial derivatives of the vector component w.r.t its axes

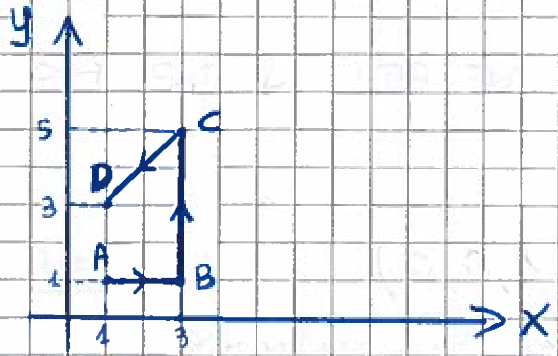
$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2 z) =$$

$$= 6x + x^2$$

$$\nabla \cdot \vec{E}\Big|_P = 6 \cdot 2 + 2^2 = 12 + 4 = 16$$

EX 4

GIVEN A VECTOR FIELD $\vec{F} = 3\,\vec{u}_y$, CALCULATE THE LINE INTEGRAL ALONG THE PATH INDICATED IN PICTURE.



A (1, 1)
B (3, 1)
C (3, 5)
D (1, 3)

$$\int_{A\to D} \vec{F}\cdot d\vec{\ell} = \int_A^B \vec{F}\cdot d\vec{\ell} + \int_B^C \vec{F}\cdot d\vec{\ell} + \int_C^D \vec{F}\cdot d\vec{\ell} =$$

THE INTEGRATION VARIABLE NEEDS TO COINCIDE WITH THE MOVING DIRECTION, IN FACT LINES HAS A DIRECTION

$$\vec{F} = F_x\,\vec{u}_x + F_y\,\vec{u}_y$$

$$d\vec{\ell} = dx\,\vec{u}_x + dy\,\vec{u}_y$$

$$= \int_{x=1}^{x=3} \underset{0}{\underset{\parallel}{F_x}}\,dx + \int_{y=1}^{y=5} F_y\,dy + \int_C^D \left(\underset{0}{\underset{\parallel}{F_x}}\,dx + F_y\,dy\right) =$$

$$= \int_1^5 3\,dy + \int_5^3 3\,dy = 3\,[y]_1^5 + 3\,[y]_5^3 =$$

$$= 3\,(5-1) + 3\,(3-5) =$$

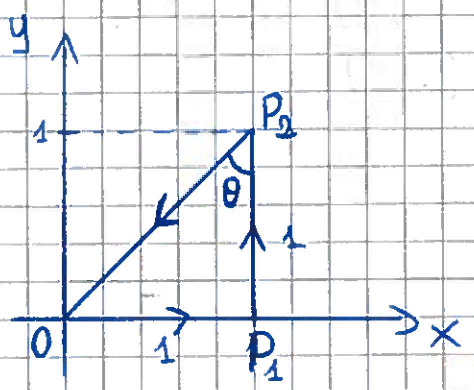
$$= 3\cdot 4 - 3\cdot 2 = 6$$

EX 5

WE CONSIDER A SCALAR FIELD:

$f(x,y,z) = 2xy + 3$ AND THE RELATION $\vec{F} = \nabla f$,

VERIFY THAT $\oint_C \vec{F} \cdot d\vec{l} = 0$ FOR THE PATH IN THE FIGURE:



$P_1 (1,0,0)$

$P_2 (0,1,0)$

We find that $\vec{F} = \frac{\partial f}{\partial x}\vec{u}_x + \frac{\partial f}{\partial y}\vec{u}_y + \frac{\partial f}{\partial z}\vec{u}_z = 2y\,\vec{u}_x + 2x\,\vec{u}_y$.

$$\oint_C \vec{F} \cdot d\vec{l} = \underbrace{\int_0^{P_1} \vec{F} \cdot d\vec{l}}_{\text{I}} + \underbrace{\int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}}_{\text{II}} + \underbrace{\int_{P_2}^{0} \vec{F} \cdot d\vec{l}}_{\text{III}}$$

I $\quad d\vec{l} = dx\,\vec{u}_x$

$$\int_0^{P_1} (F_x \vec{u}_x + F_y \vec{u}_y) \cdot dl\,\vec{u}_x = \int_0^{P_1} F_x\,dx = \int_{x=0}^{x=1} 2y\,dx = 2\overset{y=0}{y}\,[x]_0^1 = 0$$

II $\quad d\vec{l} = dy\,\vec{u}_y$

$$\int_{P_1}^{P_2} F_y\,dy = \int_{y=0}^{y=1} 2x\,dy = 2\overset{1}{x}\,[y]_0^1 = 2 \cdot 1 \cdot (1-0) = 2$$

III $\quad$ 1ST METHOD (The one seen before)

$$\int_{P_2}^{0} (F_x\,dx + F_y\,dy) = \int_{P_2}^{0} F_x\,dx + \int_{P_1}^{0} F_y\,dy = \int_{x=1}^{0} 2y\,dx + \int_{y=1}^{0} 2x\,dy =$$

$=$ I KNOW THAT THIS STRAIGHT LINE IS THE BISECTOR OF $1^{ST}/4^{TH}$ QUARTER, SO $y = x$ $\displaystyle = \int_{x=1}^{0} 2x\,dx + \int_{y=1}^{0} 2y\,dy =$

$$= \cancel{2}\cdot\left[\frac{x^2}{\cancel{2}}\right]_1^0 + 2\left[\frac{y^2}{2}\right]_1^0 = -1 - 1 = -2$$

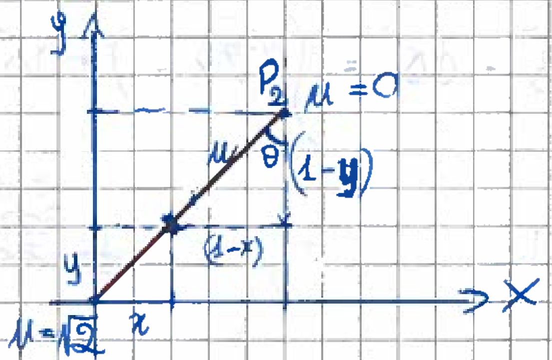
2nd METHOD

Another possibility to solve the integral III is to do the integration along the line, so not dividing the contributions along x and y:

Now I want to express x and y ~~as~~ a function of a new variable u, that goes from 0 to L.

$$\theta = \arctan\left(\frac{1}{1}\right) = 45^\circ$$

$$L = \sqrt{1+1} = \sqrt{2}$$



!

$$\begin{cases} \sin\theta = \dfrac{1-x}{u} & x = 1 - u\sin\theta & \dfrac{dx}{du} = -\sin\theta & dx = -\sin\theta\,du \\ \cos\theta = \dfrac{1-y}{u} & y = 1 - u\cos\theta & \dfrac{dy}{du} = -\cos\theta & dy = -\cos\theta\,du \end{cases}$$

$$\int_{P_2}^{0} F_x\,dx + F_y\,dy = \int_{P_2}^{0} 2y\,dx + 2x\,dy = \int_{u=0}^{u=\sqrt{2}} 2(1-u\cos\theta)\cdot -\sin\theta\,du + 2(1-u\sin\theta)\cdot -\cos\theta\,du$$

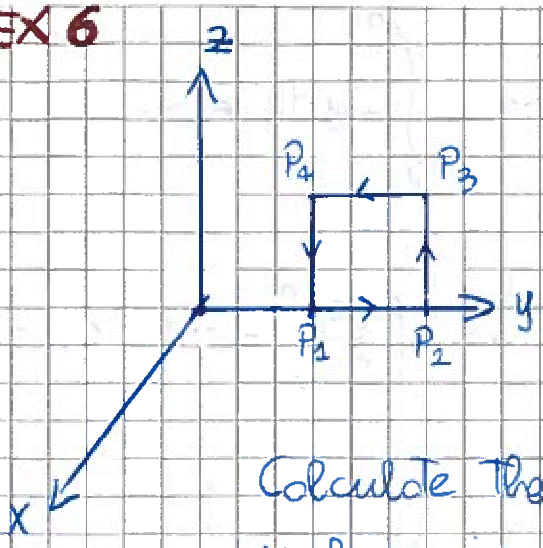
here we substitute the variables and the dx and dy

$$= \int_0^{\sqrt{2}} -\cancel{2}\left(1 - u\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{\cancel{2}}\,du - \cancel{2}\,\frac{\sqrt{2}}{\cancel{2}}\left(1 - u\frac{\sqrt{2}}{2}\right)du = -2$$

THE FIELD IS CONSERVATIVE

So, summing up, I + II + III = 0 + 2 − 2 = 0

EX 6



$P_1(0,1,0)$
$P_2(0,2,0)$
$P_3(0,2,1)$
$P_4(0,1,1)$

Calculate the flux of vector $\vec{B}$ through the surface in picture (in the positive versus of x-axis), knowing that:

$$\vec{A} = (x^2+y^2)\,\vec{u}_z$$

$$\vec{B} = \nabla \times \vec{A}$$

So the problem asks to solve:

$$\phi = \oint_S \vec{B} \cdot d\vec{s} = \int_S (\nabla \times \vec{A}) \cdot d\vec{s} \underset{\text{FOR STOKES THEOREM}}{=} \oint_C \vec{A} \cdot d\vec{e}$$

• 1^{ST} METHOD

$$\phi = \oint_C \vec{A} \cdot d\vec{e} = \underbrace{\int_{P_2}^{P_3} \vec{A} \cdot d\vec{e} + \int_{P_4}^{P_1} \vec{A} \cdot d\vec{e}}_{} = \overset{y=2}{\int_{z=0}^{1} (\cancel{x^2}+y^2)\,dz} + \overset{y=1}{\int_{z=1}^{0} (\cancel{x^2}+y^2)\,dz}$$

we already know that the movements along y axis give contributions $= 0$ because the vector $\vec{A}$ has $A_y = \emptyset$

$$= \int_{z=0}^{1} 4\,dz + \int_{z=1}^{0} dz = 4\,[z]_0^1 + [z]_1^0 = 4 - 1 = 3$$

- 2^{nd} METHOD

$$\Phi = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \int_S (2y\,\vec{u}_x - 2x\,\vec{u}_y) \cdot (dy\,dz\,\vec{u}_x) =$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & x^2+y^2 \end{vmatrix} = \vec{u}_x(2y) - \vec{u}_y(2x) = 2y\,\vec{u}_x - 2x\,\vec{u}_y$$

$$d\vec{S} = (dy \cdot dz)\,\vec{u}_x = dS\,\vec{u}_x$$

RIGHT HAND RULE

$$= \int_S 2y\,dy\,dz = 2\int_{y=1}^{2} y\,dy \int_{z=0}^{z=1} dz = \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_1^2 \cdot \left[z\right]_0^1 =$$

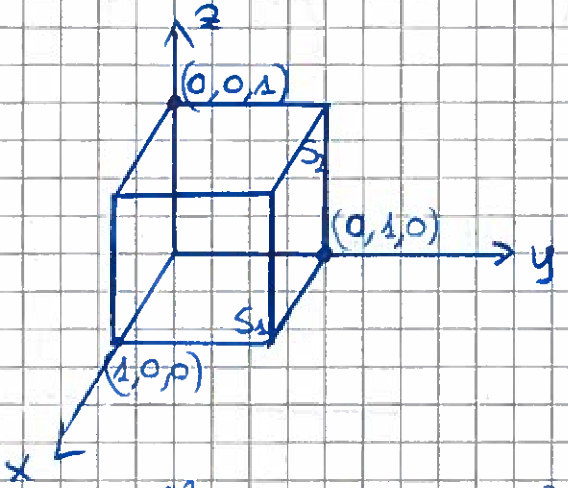
$$= (4-1) \cdot 1 = 3$$

EX 7

GIVEN A VECTOR FIELD

$$\vec{B} = (x+2)\vec{u}_x + (1-3y)\vec{u}_y + 4\vec{u}_z$$

CALCULATE THE FLUX COMING OUT FROM THE CLOSED SURFACE S



$$\phi = \oint_S \vec{B} \cdot d\vec{S} = \sum_{i=1}^{6} \phi_i$$

TO SOLVE THIS PROBLEM I have To sum the flux coming out from the 6 surfaces of the cube.

Along x-axis we have the surfaces S_1 and S_2, whose vectors are: $\pm dS\,\vec{u}_x = \pm (dy \cdot dz)\vec{u}_x$

$$\phi_1 + \phi_2 = \int_{S_1} B_x \, dy\,dz - \int_{S_2} B_x \, dy\,dz = \int_{S_1,\ x=1} (x+2)\,dy\,dz - \int_{S_2,\ x=0} (x+2)\,dy\,dz$$

$$= 3\int_0^1 dy \cdot \int_0^1 dz - 2\int_0^1 dy \cdot \int_0^1 dz = 3 - 2 = 1$$

$$\phi_3 + \phi_4 = \int_{S_3} B_y \, dx\,dz - \int_{S_4} B_y \, dx\,dz = -2 - 1 = -3$$

$$\phi_5 + \phi_6 = 4 - 4 = 0$$

So, summing up, $\phi = 1 - 3 + 0 = -2$

- Alternatively, we can use divergence Theorem: $\oint_S \vec{B} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{B})\,dV$

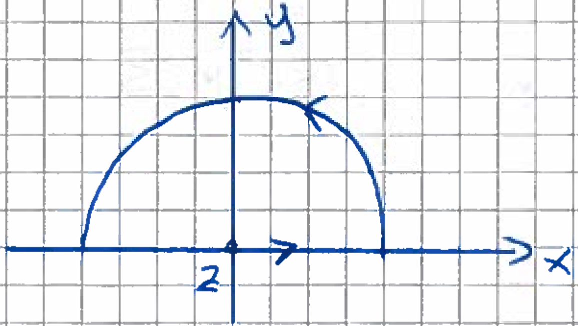
$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \boxed{1 - 3 = -2}$$

$$\rightarrow \phi = \int_V -2\,dV = -2 \cdot V = -2 \cdot 1 = -2$$

EX 8

CALCULATE THE LINE INTEGRAL $\oint_c \vec{B} \cdot d\vec{\ell}$ $\left(\vec{B} = y\, \vec{u}_x \right)$

ALONG A SEMI-CIRCLE WITH RADIUS r, USING THE STOKES THEOREM:



of course, the loop needs to be oriented!

For Stokes's Th.

$$\oint \vec{B} \cdot d\vec{\ell} = \oint_S (\nabla \times \vec{B}) \cdot d\vec{S} =$$

$$\nabla \times \vec{B} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 0 & 0 \end{vmatrix} = \vec{u}_z (0 - 1) = -\vec{u}_z$$

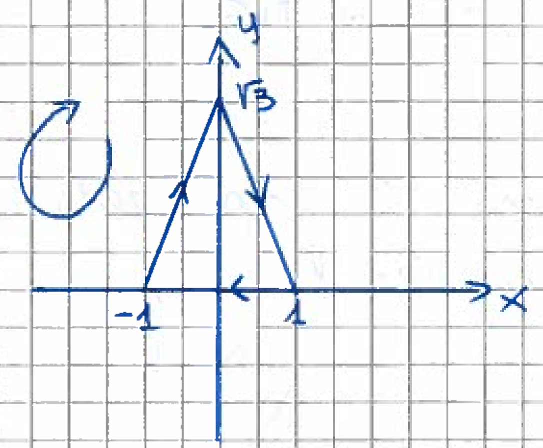
$$d\vec{S} = dS \cdot \vec{u}_z$$

RIGHT HAND RULE

$$= \oint_S -\vec{u}_z \cdot dS\, \vec{u}_z = -\int_S dS = -S = -\frac{\pi r^2}{2}$$

EX 9

GIVEN A VECTOR FIELD $\vec{C}$, CALCULATE THE INTEGRAL ALONG THE LOOP IN FIGURE: $\vec{C} = (3y)\vec{u}_x + (6x)\vec{u}_y$



We can use the Stokes Theorem:

$$\oint_C \vec{C} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{C}) \cdot d\vec{S}$$

$$\nabla \times \vec{C} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & 6x & 0 \end{vmatrix} = \vec{u}_z(6-3) = 3\vec{u}_z$$

$$d\vec{S} = dS\,\vec{u}_z$$

RIGHT HAND RULE: UNIT VECTOR IS GOING INSIDE BLACKBOARD SO IS $-\vec{u}_z$

$$\int_S 3\vec{u}_z \cdot (-dS)\vec{u}_z = \int_S -3\,dS = -3 \cdot S = -3 \cdot \left(\frac{\cancel{2} \cdot \sqrt{3}}{\cancel{2}}\right) = -3\sqrt{3}$$