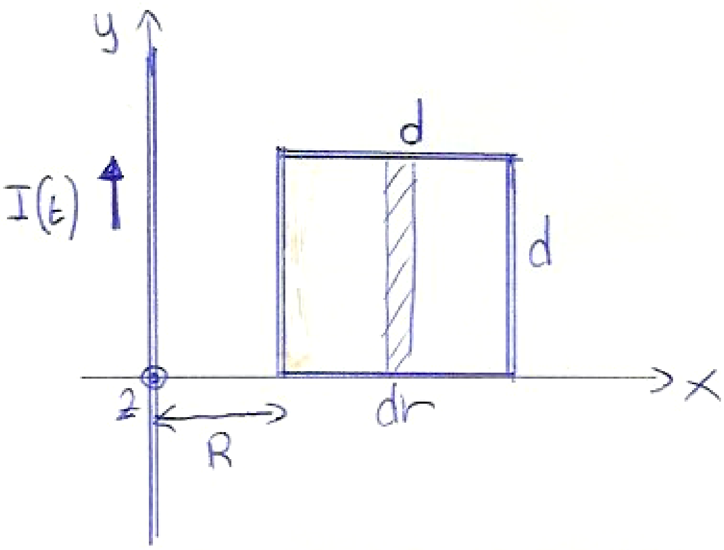
## EX 1

CALCULATE THE emf INDUCED BY A METALLIC WIRE CARRYING A CURRENT $I(t) = 0{,}5 \cos(\omega t)$ [A] ON A SQUARE COIL ($d = 20$ cm) PLACED AT DISTANCE $R = 5$ cm FROM THE WIRE.




$$\begin{pmatrix} \omega = 2\pi f \\ f = 5\text{ kHz} \end{pmatrix}$$

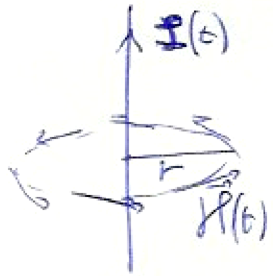
$$I(t) \xrightarrow[\text{Ampere Law}]{\text{generates}} B(t) \xrightarrow[\text{Faraday Law}]{\text{induces}} emf$$

WE CAN DECOMPOSE THE EXERCISE IN TWO STEPS:

BE CAREFULL THAT HERE THE FIELD $\vec{B}$ GENERATED BY $I(t)$ IS DIFFERENT ON THE VARIOUS POINTS OF THE COIL (BECAUSE $d > R$)

APPLYING Ampere's law:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S}$$




$$\vec{H} = H_\varphi \vec{u}_\varphi$$

$$d\vec{\ell} = r \cdot d\varphi \, \vec{u}_\varphi$$

$$H_\varphi \cdot r \cdot 2\pi = I(t)$$

$$\longrightarrow \vec{H}(t) = \frac{I(t)}{2\pi r} \vec{u}_\varphi$$

$$\vec{B}(t) = \frac{\mu I(t)}{2\pi r} \vec{u}_\varphi$$

$$= -\frac{\mu I(t)}{2\pi r} \vec{u}_z$$

• WE APPLY THE FARADAY LAW:

$$emf = \oint_C \vec{E} \cdot d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad d\vec{S} = dS\,\vec{u}_z = $$
$$= dr \cdot d\,\vec{u}_z$$

$$emf = -\oint_S \frac{\partial}{\partial t}\left(\frac{\mu I(t)}{2\pi r}\,\vec{u}_z\right) \cdot \left(d \cdot dr\,\vec{u}_z\right) =$$

$$= \int_{r=R}^{R+d} \frac{\mu \cdot d}{2\pi} \frac{\partial I(t)}{\partial t} \cdot \frac{dr}{r} =$$

$$= \frac{\mu \cdot d}{2\pi} \cdot \frac{\partial I(t)}{\partial t} \int_R^{R+d} \frac{dr}{r} =$$

$$= \frac{\mu \cdot d}{2\pi} \cdot \frac{\partial I(t)}{\partial t} \cdot \left[\ln(r)\right]_R^{R+d} =$$

$$= \frac{\mu d}{2\pi} \cdot \ln\left(\frac{R+d}{R}\right) \cdot \frac{\partial}{\partial t}\left[0{,}5 \cdot \cos(\omega t)\right] =$$

$$= \frac{\mu d}{2\pi} \cdot \ln\left(\frac{R+d}{R}\right) \cdot \left[0{,}5 \cdot \omega \cdot -\sin(\omega t)\right]$$

$$|emf| = \frac{\overset{2}{\cancel{4\pi}} \cdot 10^{-7}}{\cancel{2\pi}}\left[\frac{H}{m}\right] \cdot 0{,}2\,m \cdot \ln\left(\frac{25}{5}\right) \cdot 0{,}5 \cdot 2\pi \cdot 5 \cdot 10^3\,[Hz] \cdot \sin(\omega t)\,[A]$$

$\mu = 4\pi \cdot 10^{-7}\ \frac{H}{m}$

$$= 2 \cdot 10^{-7} \cdot 0{,}2 \cdot \ln(5) \cdot 0{,}5 \cdot \overbrace{2\pi \cdot 5 \cdot 10^3}^{\omega} \sin(\omega t) \quad \frac{H \cdot \cancel{m} \cdot A}{\cancel{m} \cdot s}$$

$R = 5\ cm$

$$= 1{,}01 \cdot 10^{-3} \sin(\omega t)\ [V] = \qquad \frac{V \cdot \cancel{s}}{\cancel{A}} \cdot \frac{\cancel{A}}{\cancel{s}}$$

$d = 20\ cm$

$\omega = 2\pi \cdot 5\ kHz$

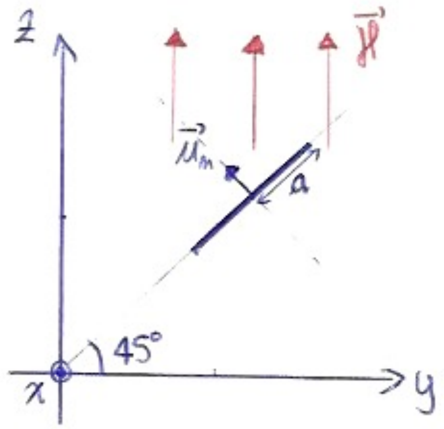
$$= \boxed{1{,}01 \cdot \sin(\omega t)\ [mV]}$$

EX 2

A CIRCULAR COIL WITH RADIUS $a$ IS PLACED IN A REGION WITH A TIME-VARYING MAGNETIC FIELD $\vec{H}(t) = 9\cos(3t)\vec{u}_z$.

THE AXIS OF THE COIL IS ORIENTED WITH AN ANGLE OF 45° WITH RESPECT TO THE $\vec{u}_z$ DIRECTION.

CALCULATE THE e.m.f. INDUCED IN THE COIL.




ACCORDING TO THE FARADAY'S LAW:

$$e.m.f = \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \cdot d\vec{s}$$

$$\vec{B} = \mu \vec{H} = \mu 9 \cos(3t)\vec{u}_z = B_z \vec{u}_z$$

$$d\vec{s} = ds\, \vec{u}_n$$

$$e.m.f. = -\frac{\partial}{\partial t}\int_S B_z \underbrace{\vec{u}_z \cdot \vec{u}_n}_{\cos 45^\circ} ds = -\frac{\partial}{\partial t}\int_S \frac{\sqrt{2}}{2} B_z\, ds =$$

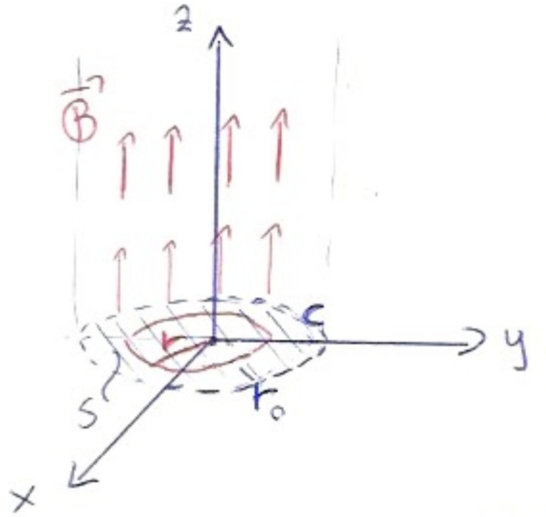
$$= -\frac{\sqrt{2}}{2} \cdot \frac{\partial}{\partial t} B_z \cdot \underbrace{\int_S ds}_{\pi a^2} = -\frac{\sqrt{2}}{2} \cdot \pi a^2 \cdot \frac{\partial}{\partial t}\left[\mu 9 \cos(3t)\right] =$$

$$= -\frac{\sqrt{2}}{2} \cdot \pi a^2 \cdot \mu 9 \cdot 3 \cdot \left[-\sin(3t)\right] = \boxed{+\frac{27 \cdot \sqrt{2}}{2} \cdot \pi a^2 \cdot \mu \sin(3t)}$$

EX 3 (FARADAY LAW)

CALCULATE THE ELECTRIC FIELD $\vec{\mathcal{E}}$ INDUCED BY A TIME-VARYING MAGNETIC FIELD $\vec{B}$ GIVEN BY:

$$\vec{B} = \begin{cases} r \cdot \sin(\omega t)\, \vec{u}_z & r \leq r_0 \\ \emptyset & r > r_0 \end{cases}$$




WE KNOW THAT A TIME-VARYING MAGNETIC FIELD IS INDUCING AN ELECTROMOTIVE FORCE e.m.f THAT IS EQUAL TO THE CIRCULATION OF $\vec{\mathcal{E}}$ OVER A CLOSED LINE.

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FARADAY LAW:

$$e.m.f = \oint_C \vec{\mathcal{E}} \cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

C: CLOSED LOOP

S: SURFACE ENCLOSED INTO THE LOOP C

THIS "FORCE" COUNTERBALANCE THE VARIATION OF THE FLUX $\vec{B}$, GENERATING ANOTHER MAGNETIC FIELD THAT VARYIES OVER THE TIME IN THE OPPOSITE SENSE.

$$\underbrace{\oint_C \vec{\mathcal{E}} \cdot d\vec{\ell}}_{\mathrm{I}} = \underbrace{-\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}_{\mathrm{II}}$$

II WE START FROM II MEMBER OF FARADAY EQUATION; DISTINGUISHING THE CASE IN WHICH $r \leq r_0$ AND $r > r_0$

SURFACE ELEMENT $d\vec{S} = dS\, \vec{u}_z = (2\pi r\, dr)\, \vec{u}_z$

$\rightarrow \boxed{r < r_0}$ $$-\int_{S_r} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\int_{S_r} \frac{\partial}{\partial t}\left[r \sin(\omega t)\right] \vec{u}_z \cdot (2\pi r\, dr)\, \vec{u}_z =$$

$$= -\int_{S_r} r \cdot \omega \cdot \cos(\omega t) \cdot 2\pi r \, dr = -\int_0^r 2\pi\omega \cos(\omega t) \cdot r^2 dr =$$

$$= -2\pi\omega \cos(\omega t) \int_0^r r^2 dr = \boxed{-2\pi\omega \cdot \frac{r^3}{3} \cos(\omega t)}$$

→ $\boxed{r > r_0}$ SINCE THE FIELD $\vec{B}$ VARYS WITH TIME ONLY FOR $r \leq r_0$, THE CONTRIBUTION TO THE FLUX OF $\frac{\partial \vec{B}}{\partial t}$ IS GIVEN ONLY BY THE FLUX CALCULATED FOR $r \leq r_0$:

$$-\int_{S_{r_0}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\int_0^{r_0} 2\pi\omega \cos(\omega t) \, r^2 dr = \boxed{-2\pi\omega \cdot \frac{r_0^3}{3} \cos(\omega t)}$$

I NOW FOR THE FIRST MEMBER OF THE FARADAY LAW WE NEED TO CALCULATE THE CIRCULATION OF $\vec{\mathcal{E}}$ OVER THE LOOP WITH RADIUS $r$.

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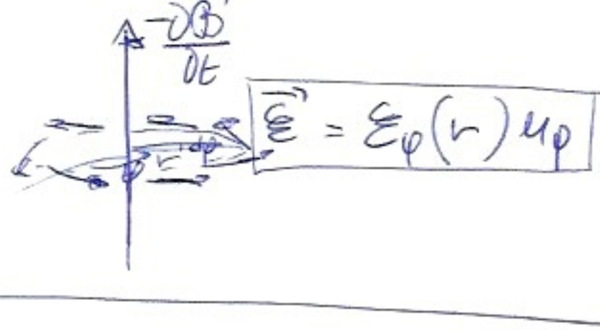
REMEMBER THAT FARADY LAW CAN BE WRITTEN AS FOLLOW:

$$\nabla \times \vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t}$$

→ IT MEANS THAT THE INDUCED FIELD $\vec{\mathcal{E}}$ HAS FIELD LINES THAT ARE CLOSED AND PERPENDICULAR TO THE VARIATION OF $\vec{B}$ OVER TIME:

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SO:

$$\oint_C \vec{\mathcal{E}} \cdot d\vec{\ell} = \oint_C (\mathcal{E}_\varphi \vec{u}_\varphi) \cdot (r \cdot d\varphi \, \vec{u}_\varphi) = \int_0^{2\pi} \mathcal{E}_\varphi r \, d\varphi = \mathcal{E}_\varphi \cdot r \cdot 2\pi$$

EQUALLING THE TWO MEMBERS OF FARADAY LAW WE GET:

$\boxed{r < r_0}$

$$\mathcal{E}_\varphi \cancel{r} \cdot \cancel{2\pi} = -\cancel{2\pi}\,\omega \frac{r^{\cancel{3}\,2}}{3} \cos(\omega t)$$

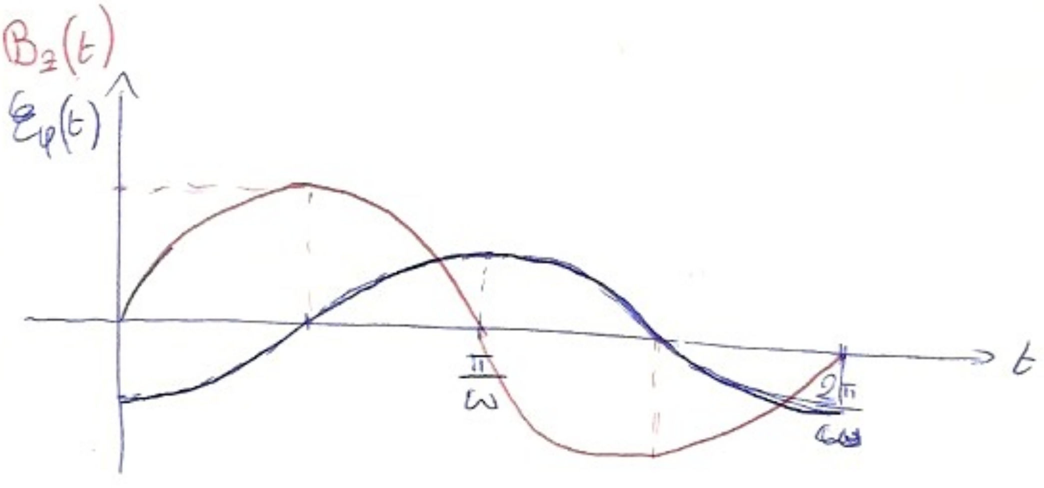
$$\mathcal{E}_\varphi = -\frac{\omega r^2}{3} \cos(\omega t)$$

$\boxed{r > r_0}$

$$\mathcal{E}_\varphi \cdot r \cdot \cancel{2\pi} = -\cancel{2\pi}\,\omega \frac{r_0^3}{3} \cos(\omega t)$$

$$\mathcal{E}_\varphi = -\frac{\omega r_0^3}{3r} \cos(\omega t)$$

$$\vec{\mathcal{E}} = \begin{cases} -\dfrac{\omega r^2}{3} \cos(\omega t)\, \vec{u_\varphi} \\ \\ -\dfrac{\omega r_0^3}{3r} \cos(\omega t)\, \vec{u_\varphi} \end{cases}$$




$\omega = 2\pi f \quad f = \frac{\omega}{2\pi}$

$T = \frac{1}{f} = \frac{2\pi}{\omega}$

WE FOUND THAT WHEN

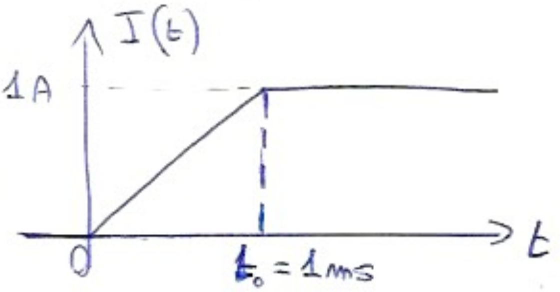
$$\begin{cases} \dfrac{\partial B}{\partial t} > 0 \implies \vec{\mathcal{E}} \text{ IS NEGATIVE} \\ \\ \dfrac{\partial B}{\partial t} < 0 \implies \vec{\mathcal{E}} \text{ IS POSITIVE} \end{cases}$$

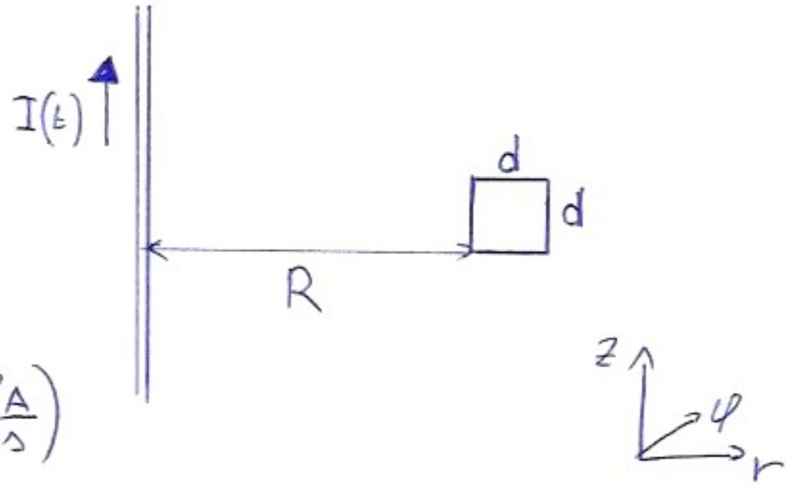
THIS BECAUSE e.m.f GENERATES A MAGNETIC FLUX THAT COUNTERACTS THE VARIATION OF ORIGINAL MAGNETIC FLUX

EX4

A STRAIGHT ELECTRIC WIRE CARRYING A CURRENT $I(t)$ IS PLACED AT DISTANCE $R = 1$ cm FROM A SMALL SQUARE COIL WITH SIZE $d = 1$ mm.

CALCULATE THE emf INDUCED IN THE COIL IF $I(t)$ IS GIVEN BY:




$$I(t) = \begin{cases} K_0 \cdot t & t \le t_0 \\ 1\ \text{A} & t > t_0 \end{cases} \qquad \left(K_0 = 10^3 \frac{\text{A}}{\text{s}}\right)$$

WE CAN "DECOMPOSE" THE PROBLEM INTO TWO STEP:

$$I(t) \xrightarrow[\text{Ampere law}]{\text{GENERATES}} B(t) \xrightarrow[\text{Faraday law}]{\text{GENERATES}} e.m.f.$$

THE TIME-VARYING CURRENT, FLOWING THROUGH THE WIRE, IS GENERATING A TIME-VARYING MAGNETIC FIELD $\vec{H}(t)$, THAT FOR SYMMETRICAL REASON IS: $\vec{H}(t) = H_\varphi(r)\,\vec{u}_\varphi$

Ampere law $\displaystyle \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{s}$

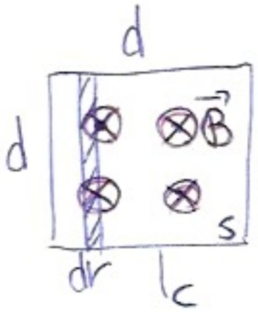
$$H_\varphi \cdot 2\pi \cdot R = I(t)$$

$$\boxed{\vec{H}(t, R) = \frac{I(t)}{2\pi R}\,\vec{u}_\varphi}$$

SINCE THE COIL IS MUCH SMALLER THAN R ($d \ll R$) WE CAN APPROXIMATE THAT THE MAGNETIC FIELD $\vec{H}$ THAT FILLS THE COIL AREA IS CONSTANT $\forall t$.

$$\vec{B} = \mu \vec{H} = \frac{\mu I(t)}{2\pi R}\vec{u}_\varphi$$

SO WE FOUND THAT THE MAGNETIC FLUX DENSITY $\vec{B}$ IS ENTERING INTO THE COIL PLANE.




FROM "Faraday law"

$$e.m.f = \oint_C \vec{E}\cdot d\vec{\ell} = -\int_S \frac{\partial}{\partial t}\vec{B}(t)\cdot d\vec{S}$$

$$d\vec{S} = dS(+\vec{u}_\varphi) = d\cdot dr\,\vec{u}_\varphi$$

$$e.m.f = -\int_S \left(\frac{\partial}{\partial t}B_\varphi\,\vec{u}_\varphi\right)\cdot\left(d^2\vec{u}_\varphi\right) = -\int_0^d \frac{\partial}{\partial t}\left[\frac{\mu I(t)}{2\pi R}\right]\cdot d\cdot dr =$$

$$= -\left[\frac{\partial}{\partial t}I(t)\right]\cdot\frac{\mu\cdot d^2}{2\pi R} =$$

$$= \begin{cases} -\frac{\partial}{\partial t}\left[K_0 t\right]\frac{\mu d^2}{2\pi R} = -\frac{K_0\,\mu\cdot d^2}{2\pi R} & (t < t_0) \\ \emptyset & (t > t_0) \end{cases}$$

If $t < t_0$ we get:

$$|emf| = 10^3\,\frac{A}{s}\cdot \overset{2}{\cancel{4\pi}}\cdot 10^{-7}\,\frac{H}{m}\cdot\frac{(10^{-3}\,m)^2}{\cancel{2\pi}\cdot 10^{-2}\,m} =$$

$$= \frac{10^3\cdot 2\cdot 10^{-7}\cdot 10^{-6}}{10^{-2}}\cdot\frac{A}{s}\cdot\frac{H}{\cancel{m}}\cdot\frac{\cancel{m^2}}{\cancel{m}} = 2\cdot 10^{-8}\,\frac{\cancel{A}}{\cancel{s}}\cdot\frac{V\cdot\cancel{s}}{\cancel{A}} = \boxed{2\cdot 10^{-8}\,V}$$